Statistical properties of passive tracers in a positive four-point vortex model

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Stochastic properties of systems formed by many passive particles conducted by four point vortices, each one with positive circulation, are investigated. A statistical χ^2 test is developed in order to study the spatial distribution of particles in the chaotic background ($\lambda_L > 0$). The fact that the uniform distribution is an invariant measure of the spatial distribution of particles is used to debug the χ^2 test. This procedure is applied in the same conditions as described in Babiano *et al.* in order to study the uniformity of passive particles. It is observed that uniformity is not attained up to times of order 10^5 natural time unity, when either a Gaussian or a uniform initial distribution is considered in a small region away from the vortices.

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I. INTRODUCTION

The advection of a passive tracer in a hydrodynamical flow is an important issue in oceanography and atmospheric physics [1–4]. The Eulerian and Lagrangian approaches of hydrodynamic problems are strictly connected by the ordinary differential equation [5]:

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x},t), \tag{1}$$

supplemented by a given initial condition $\mathbf{x}(\mathbf{0})$. Here, $\mathbf{v}(\mathbf{x},t)$ is the Eulerian velocity field and $\mathbf{x}$ is the position of a fluid particle (Lagrangian point of view). The dynamics of the Eulerian velocity field, governed by the Navier-Stokes or Euler equations, usually is reduced, after a Galerkin truncation, to a system of coupled nonlinear differential equations

$$\dot{\hat{\mathbf{v}}} = \mathbf{F}(\hat{\mathbf{v}},t), \tag{2}$$

which represents the time evolution of the Fourier components list $\hat{\mathbf{v}} = \{\hat{\mathbf{v}}_{\mathbf{k}}\}_{k \leq K_{max}}$ of the velocity field $\mathbf{v}(\mathbf{x},t) = \Sigma_{\mathbf{k}\in\mathbf{Z}^3}\hat{\mathbf{v}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$. The Lyapunov exponents associated with Eqs. (1) and (2) are, respectively, known as the Lagrangian and Eulerian Lyapunov exponents. Denote them by λ_L and λ_E. The cases corresponding to $\lambda_L > 0$ ($\lambda_E > 0$) are known as Lagrangian (Eulerian) chaoticity. These two kinds of chaoticity are disconnected and they are studied in [1]. (See also [4].)

Our goal is to investigate the distribution of passive tracers advected by the point vortices of the two-dimensional Euler equation in the region $\lambda_L > 0$ [1].

This Brief Report is organized as follows. In Sec. II we describe the mathematical setup. In Sec. III we report the results of numerical experiments performed on systems formed by many passive particles conducted by $N=4$ point vortices, each one with positive circulation. An Appendix is added at the end, which includes the main idea of the statistical method that has been used.

II. MATHEMATICAL FRAMEWORK

The two-dimensional Euler equation reads

$$\partial_t \omega + J(\omega, \nabla^{-2}\omega) = 0, \tag{3}$$

where $\omega = \omega(t;x,y)$ denotes the vorticity field, J is the two-dimensional Jacobian, and ∇^{-2} stands for the inverse of the two-dimensional Laplacian. It is known that Eq. (3) possesses singular solutions of the form [6]

$$\omega(t;x,y) = \sum_{\alpha=1}^{N} k_\alpha \delta[x - x_\alpha(t)]\delta[y - y_\alpha(t)]. \tag{4}$$

Here, $\delta(\bullet)$ denotes the Dirac δ function, $(x_\alpha(t), y_\alpha(t))$ represents a point vortex in the unbounded plan with circulation k_α, and N is the total number of point vortices, and their time evolution has the following Hamiltonian structure:

$$k_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha},$$

$$k_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \tag{5}$$

where the Hamiltonian H is given by

$$H \equiv H(x_1, \ldots, x_N; y_1, \ldots, y_N)$$
$$= -\frac{1}{8\pi} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} k_\alpha k_\beta \ln[(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2]. \tag{6}$$

Introducing the *complex* point vortex ($i = \sqrt{-1}$):

$$z_\alpha(t) = x_\alpha(t) + i y_\alpha(t), \tag{7}$$

after some straightforward calculation, one can show that Eq. (5) is equivalent to the *complex* equations:

$$\dot{z}^*_\alpha = \frac{1}{2\pi i} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{k_\beta}{z_\alpha - z_\beta}, \quad \alpha = 1,2,\ldots,N, \tag{8}$$

where the symbol * denotes complex conjugation.

By definition, a passive tracer is a point vortex with zero circulation. Ergo, the equation for a passively advected tracer $z(t) = x(t) + iy(t)$, initially located at $z(0)$, and driven by the vortices defined by Eq. (8) is

$$\dot{z}^* = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{k_\beta}{z - z_\beta}. \tag{9}$$

III. SIMULATION RUNS AND RESULTS

This paper reports the results of numerical experiments performed on systems formed by many passive particles conducted by four-point vortices, each one with positive circulation. Equations (8) and (9) have been integrated by a fourth-order Runge-Kutta method with time step $\Delta t = 5 \times 10^{-3}$ n.t.u. (natural time unity). In order to perform this study, three different simulations were run, each one using a set of 10^4 particles advected by a four-point vortex originally situated at points $(5,0),(0,5),(-5,0),(1,-4)$. All these vortices have circulation equal to 10.

In the first two experiments, particles were generated away from the vortices, using either Gaussian or a uniform distribution in a small square of the form

$$[c_x - \epsilon; c_x + \epsilon] \times [c_y - \epsilon; c_y + \epsilon], \tag{10}$$

with $(c_x, c_y) = (8,8)$ and $\epsilon = 1/5$. In the third experiment, particles were generated uniformly in the circular spot observed as described by Babiano *et al.* [1]. It is well known from the literature that a uniform distribution is an invariant measure of this dynamics. This fact is used to test the procedure described in this paper.

As it is expected, in all experiments one observes that the particles spread in a *circular* spot surrounding the four regular islands $(\lambda_L = 0)$. Our purpose is to find out whether these particles could be considered uniformly distributed or not on the chaotic background defined by $\lambda_L > 0$.

A statistical χ^2 test has been applied to this spot excluding the regular islands $\lambda_L = 0$. In order to perform this test, a set of small circles with radius 0.5 (in length natural units) was generated randomly inside the spot. The statistical test is based on the difference between a theoretically uniform distribution in the small circles and the simulated distribution (the situation observed), by means of the computation of a distance d and the probability that this distance takes large values (see the Appendix). The results given by this test at different times between $t = 2\times 10^3$ and $t = 10^5$ n.t.u., with time step equal to 2×10^3, in terms of the probability of overcrossing the observed distance, are plotted in Fig. 1 in a linear-logarithmic scale.

The results of the χ^2 test are the following: in the first two experiments, the uniformity of the distributions of the particles, in the condition mentioned above, was not achieved up to $t = 10^5$ n.t.u., as illustrated in Fig. 1 (uniform distribution, long dashed line; Gaussian distribution, dotted line) [7]. The third experiment is plotted in dashed line, which tells us that there are no reasons to reject the hypothesis that the uniform distribution of the particles is not changed with time.

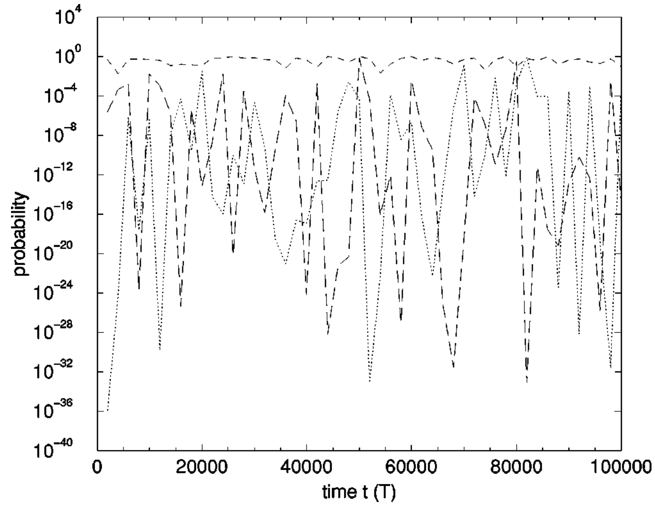

FIG. 1. Probability of overcrossing the observed distance according to the χ^2 distribution. Uniform initial distribution in a small square: long dashed line. Gaussian initial distribution around the point (8,8): dotted line. The dashed line validates the χ^2 test.

In conclusion, we have performed computational simulations on systems formed by 10^4 passive particles advected by the action of four vortices. Our goal was to get indications about the statistical distribution that a naive visual inspection of the figures plotted in Babiano *et al.* [1] suggests to be uniform. We conclude that, if uniformity is to be achieved, it will take a very long time.

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APPENDIX: THE χ^2 TEST

This is a well-known test from the statisticians—see [8], for instance, for the one-dimensional version. A two-dimensional version of the test will be the content of this appendix, since the problem under study corresponds to a particular situation that, as far as we know, has not been considered in the literature and is not well covered by methods like those described in [9]. However, the idea of the χ^2 goodness of fit test that we present here is the same.

The region of the plane where the particles evolve, let us denote it by R, is considered in the test. Given the possible complexity of the configuration of this region we considered that the islands created by the vortices were approximately circular shaped. Then, under the assumption that the distribution of particles is uniform, considering a total of N particles in R, if m circles are generated in this region (with random centers) the number of particles that fall inside each circle should be proportional to the area of the circle intercepted with the region R. Thus, a sample of circles can be generated and its interceptions with the region R, denoted by C_i, $i = 1,2,\ldots,m$, are considered in the test. If n_i is the number of particles in the small subregion C_i, n_{m+1} is the number of particles that fall outside all subregions C_i, $p_i = \text{area}(C_i)/\text{area}(R)$, for $i=1,2,\ldots,m$ and $p_{m+1}=1-\Sigma_{i=1}^{m} p_i$, then we consider

$$d=\sum_{i=1}^{m+1} \frac{(n_i-Np_i)^2}{Np_i}$$

the square distance between the families $(n_i)_i$ and $(Np_i)_i$ with $i=1,2,\ldots,m+1$. If the uniformity of the distribution in the region R is true, then d should be approximately a realization of a χ^2 distributed random variable with m degrees of freedom, if one assumes that the center and radius of R as well as the islands are known. This argument enables the construction of a hypothesis test of χ^2 type. The assumption of uniformity for the distribution of the particles in the fixed region R will be rejected, if the distance obtained in the simulation is greater than the critical value, that is, the probability of overcrossing the observed distance is, in that case, very small. Otherwise, we will accept the uniformity hypothesis.

In the work presented in this Brief Report, a slightly different version of this test has been adopted, in order to reduce complexity of the computations. The modification is as follows. If n_i is the number of particles in the small subregion C_i, and N is the total number of particles captured inside all subregions C_i, and $p_i = \text{area }(C_i)/\text{area }(R)$, for $i=1,2,\ldots,m$, then it would be reasonable to assume that the square distance between the families $(n_i)_i$ and $(Np_i)_i$ where $i=1,2,\ldots,m$,

$$d=\sum_{i=1}^{m} \frac{(n_i-Np_i)^2}{Np_i},$$

should be approximately a realization of a χ^2 distributed random variable with $m-1$ degrees of freedom, if one assumes that the center and radius of R as well as the islands are known. According to this argument, the assumption of uniformity for the distribution of the particles in the fixed region R will be rejected if the distance obtained in the simulation is greater than the critical value. Otherwise, we will accept the uniformity hypothesis.

In analogy to the conventional statistical goodness-of-fit χ^2 test, the adoption of very small circles should be avoided. For this reason, only subregions where $Np_i \geqslant 5$ will be considered.

In order to verify the behavior of the test described above, 1000 samples have been simulated in the region R with the same parameters as defined by our problem and the same number of particles, 10 000, using a uniform distribution on R. The square distance d has been computed for each simulation and it has been verified that the values of d do fit a χ^2 distribution with $m-1$ degrees of freedom. The type-I error of the test, i.e., the probability of making the wrong decision in rejecting the uniformity hypothesis when the uniformity is a fact, is controlled and the 5% significance level is attained. Investigation of the type-II error, i.e., the probability of making the wrong decision in accepting uniformity when it does not correspond to reality, was conducted, letting us believe that this error behaves well enough. Due to the complexity of the region R, we did not go further in this line of study.

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